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Applying Gröbner bases to solve reduction problems for Feynman integrals

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Abstract: We describe how Gröbner bases can be used to solve the reduction problem for Feynman integrals, i.e. to construct an algorithm that provides the possibility to express a Feynman integral of a given family as a linear combination of some master integrals. Our approach is based on a generalized Buchberger algorithm for constructing Gröbner-type bases associated with polynomials of shift operators. We illustrate it through various examples of reduction problems for families of one- and two-loop Feynman integrals. We also solve the reduction problem for a family of integrals contributing to the three-loop static quark potential.

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1. Introduction

The important mathematical problem of evaluating Feynman integrals arises naturally in elementary-particle physics when one treats quantum-theoretical amplitudes in the framework of perturbation theory. This problem originated in the early days of perturbative quantum field theory. Over more than five decades, a great variety of methods for evaluating Feynman integrals has been developed. However, to check whether the Standard Model or its extensions describe adequately particle interactions observed in experiments, one needs to perform more and more sophisticated calculations, so that one tries not only to update existing methods but also develop new effective methods of evaluating Feynman integrals.

After a tensor reduction based on some projectors a given Feynman graph generates various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators which we shall also call *indices*. Let $F(a_1, a_2, \ldots, a_n)$ be a scalar dimensionally regularized [1] Feynman integral corresponding to a given graph and labelled by the (integer) indices, a_i:

$$F(a_1, \ldots, a_n) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_n^{a_n}}, \tag{1.1}$$

where k_i, $i = 1, \ldots, h$, are loop momenta and the denominators are given by

$$E_r = \sum_{i \geq j \geq 1} A_r^{ij}\, p_i \cdot p_j - m_r^2, \tag{1.2}$$

with $r = 1, \ldots, n$. The matrix A_r^{ij} depends on the choice of the loop momenta. The momenta p_i are either the loop momenta $p_i = k_i$, $i = 1, \ldots, h$, or independent external momenta $p_{h+1} = q_1, \ldots, p_{h+n} = q_N$ of the graph. Irreducible polynomials in the numerator can be represented as denominators raised to negative powers. For example, the

denominator corresponding to the propagator of a massless particle is $k^2 = k_0^2 - \vec{k}^2$. Usual prescriptions $k^2 = k^2 + i0$, etc. are implied. Formally, dimensional regularization [1] is denoted by the change $\mathrm{d}^4k = \mathrm{d}k_0\mathrm{d}\vec{k} \to \mathrm{d}^dk$, where $d = 4 - 2\varepsilon$ is a general complex number. The Feynman integrals are functions of the masses, m_i, and kinematic invariants, $q_i \cdot q_j$. However, we shall omit this dependence because we shall pay special attention to the dependence on the indices. We shall also omit the dependence on d.

A straightforward strategy is to evaluate, by some methods, every scalar Feynman integral resulting from the given graph. If the number of these integrals is small this strategy is quite reasonable. In non-trivial situations, where the number of different scalar integrals can be at the level of hundreds and thousands, this strategy looks too complicated. A well-known optimal strategy here is to derive, without calculation, and then apply some relations between the given family of Feynman integrals as *recurrence relations*. A well-known standard way to obtain such relations is provided by the method of integration by parts (IBP) [2] which is based on the fact that any dimensionally regularized integral of the form

$$\int \mathrm{d}^dk_1\mathrm{d}^dk_2 \ldots \frac{\partial f}{\partial k_i^\mu} \tag{1.3}$$

is equal to zero. Here f is the integrand in (1.1). More precisely, one tries to use IBP relations in order to express a general dimensionally regularized integral of the given family as a linear combination of some 'irreducible' integrals which are also called *master* integrals. Therefore the whole problem decomposes into two parts: the construction of a reduction algorithm and the evaluation of the master Feynman integrals.

There were several recent attempts to make the reduction procedure systematic:

(*i*) Using the fact that the total number of IBP equations grows faster than the number of independent Feynman integrals, when one increases the total power of the numerator and denominator, one can sooner or later obtain an overdetermined system of equations [3, 4] which can be solved. (There is a public version of implementing the corresponding algorithm on a computer [5].)

(*ii*) Using relations that can be obtained by tricks with shifting dimension [6].

(*iii*) Baikov's method [7].

Another attempt in this direction is based on the use of Gröbner bases [9]. The first attempt to apply the theory of Gröbner bases in the reduction problems for Feynman integrals was made in [10], where IBP relations were reduced to differential equations. To do this, it is assumed that there is a non-zero mass for each line. For differential equations one can then apply some standard algorithms for constructing corresponding Gröbner bases.

In [10, 11] it was pointed out that the straightforward implementation of the Buchberger algorithm in the case of IBP relations is problematic because it requires a lot of computer time even in simple examples. One of the possible modifications is related to the Janet bases [12]. We are going to modify the Buchberger algorithm in another way, taking into account explicitly such properties as boundary conditions (which characterize all the regions of indices where the Feynman integrals are equal to zero), so that it will be possible to apply it to solve the reduction problem in complicated situations.

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In the next section, we shall briefly describe what the Gröbner basis and the Buchberger algorithm are in the classical problem related to solving systems of algebraic equations. In section 3, we shall turn to IBP relations and describe our strategy of constructing Gröbner bases associated with the given problem, with the help of a modification of the standard Buchberger algorithm. We shall explain how these results can be applied to solve IBP relations. We shall illustrate our strategy, through various examples, in section 4. In particular, we shall apply our algorithm to a family of three-loop Feynman integrals with a one-loop insertion relevant to the three-loop quark potential. We shall also evaluate the master integrals using the method based on Mellin-Barnes representation. In Conclusion, we shall characterize the status and perspectives of our method.

2. Gröbner basis and Buchberger algorithm

The notion of the Gröbner basis was invented by Buchberger [9] when he constructed an algorithm to answer certain questions on the structure of ideals of polynomial rings.

Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$ be the commutative ring of polynomials of n variables $x_1, \ldots, x_n$ over $\mathbb{C}$ and $\mathcal{I} \subset \mathcal{A}$ be an ideal[1]. A classical problem is to construct an algorithm that shows whether a given element $g \in \mathcal{A}$ is a member of $\mathcal{I}$ or not. A finite set of polynomials in $\mathcal{I}$ is said to be a *basis* of $\mathcal{I}$ if any element of $\mathcal{I}$ can be represented as a linear combination of its elements, where the coefficients are some elements of $\mathcal{A}$. Let us fix a basis $\{f_1, f_2, \ldots, f_k\}$ of $\mathcal{I}$. The problem is to find out whether there are polynomials $r_1, \ldots, r_k \in \mathcal{A}$ such that $g = r_1 f_1 + \ldots + r_k f_k$.

Let $n = 1$. In this case any ideal is generated by one element $f = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$. Now if we want to find out whether an element $g = b_0 + b_1 x + b_2 x^2 + \ldots + b_l x^l$ can be represented as rf we first check if $l \geq m$. If so, we replace g with $g - (b_l/a_m)x^{l-m} f$, 'killing' the leading term of g. This procedure is repeated until the degree of a 'current' polynomial obtained from g becomes less than m. It is clear that the resulting polynomial is equal to zero if and only if g can be represented as rf.

Now let $n > 1$. Let us consider an algorithm that will answer this problem for some bases of the ideal. (We will see later that this problem can be solved if we have a so-called Gröbner basis at hand.) To describe it, one needs the notion of an *ordering of monomials* $cx_1^{i_1} \ldots x_n^{i_n}$ where $c \in \mathbb{C}$ and the notion of the *leading term* (an analogue of the intuitive one in the case $n = 1$). In the simplest variant of *lexicographical* ordering, a set $(i_1, \ldots, i_n)$ is said to be *higher* than a set $(j_1, \ldots, j_n)$ if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, $\ldots$, $i_{l-1} = j_{l-1}$ and $i_l > j_l$. The ordering is denoted as $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$. We shall also say that the corresponding monomial $cx_1^{i_1} \ldots x_n^{i_n}$ is higher than the monomial $c' x_1^{j_1} \ldots x_n^{j_n}$.

One can introduce various orderings, for example, the *degree-lexicographical* ordering, where $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering. The only two axioms that the

[1] A non-empty subset $\mathcal{I}$ of a ring R is called a left (right) ideal if (i) for any $a, b \in \mathcal{I}$ one has $a + b \in \mathcal{I}$ and (ii) for any $a \in \mathcal{I}, c \in R$ one has $ca \in \mathcal{I}$ ($ac \in \mathcal{I}$ respectively). In the case of commutative rings there is no difference between left and right ideals.

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ordering has to satisfy are that 1 is the only minimal element under this ordering and that if $f_1 \succ f_2$ then $gf_1 \succ gf_2$ for any g.

Let us fix an ordering. The *leading term* (under this ordering) of a polynomial

$$P(x_1, \ldots, x_n) = \sum c_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

is the non-zero monomial $c_{i_1^0,\ldots,i_n^0} x_1^{i_1^0} \ldots x_n^{i_n^0}$ such that the degree $(i_1^0, \ldots, i_n^0)$ is higher than the degrees of other monomials in P. Let us denote it by $\hat{P}$. We have $P = \hat{P} + \tilde{P}$, where $\tilde{P}$ is the sum of the remaining terms.

Let us return to the problem formulated above. Suppose that the leading term of the given polynomial g is divisible by the leading term or some polynomial of the basis, i.e. $\hat{g} = Q\hat{f}_i$ where Q is a monomial. Let $g_1 = g - Qf_i$. It is clear that the leading term of g_1 is lower than the leading term of g and that $g_1 \in \mathcal{I}$ if and only if $g \in \mathcal{I}$. One can go further and proceed with g_1 as with g, using the same f_i or some other element f_j of the initial basis, and obtain similarly $g_2, g_3, \ldots$. The procedure is repeated until one obtains $g_l \equiv 0$ or an element g_l such that $\hat{g_k}$ is not divisible by any leading term $\hat{f}_i$. We will say that g is reduced to g_l modulo the basis $\{f_1, f_2, \ldots, f_k\}$.

A basis $\{f_1, f_2, \ldots, f_k\}$ is called a *Gröbner basis* of the given ideal if any polynomial $g \in \mathcal{I}$ is reduced by the described procedure to zero for any sequence of reductions. Given a Gröbner basis we obtain an algorithm to verify whether an element $g \in \mathcal{A}$ is a member of $\mathcal{I}$. There are many other questions on the structure of the ideal that can be answered constructively if one has a Gröbner basis, but they are beyond the topic of the paper.

Generally a basis is not a Gröbner basis. Let $f_1 = x_1$ and $f_2 = 1 + x_2^2$ and let $\mathcal{I}$ be generated by f_1 and f_2. It is easy to verify that $\{f_1, f_2\}$ is a Gröbner basis of $\mathcal{I}$. Now let $f_1' = x_1 x_2$. The set $\{f_1', f_2\}$ is again a basis of $\mathcal{I}$ (indeed, $f_1' = x_2 f_1$ and $f_1 = -x_2 f_1' + x_1 f_2$). However, $\{f_1', f_2\}$ is not a Gröbner basis because the element $x_1 \in \mathcal{I}$ cannot be reduced modulo $\{f_1', f_2\}$.

On the other hand, given any initial basis $\{f_1, f_2, \ldots, f_k\}$ of the ideal $\mathcal{I}$ one can construct a Gröbner basis starting from it and using the *Buchberger algorithm* which consists of the following steps.

Suppose that $\hat{f}_i = wq_i$ and $\hat{f}_j = wq_j$ where w, q_i and q_j are monomials and w is not a constant. Define $f_{i,j} = f_i q_j - f_j q_i$. Reduce this polynomial modulo the set $\{f_i\}$ as described above. If one obtains a non-zero polynomial by this reduction, add it to the set $\{f_i\}$. Consider then the other elements with $\hat{f}_i' = wq_i'$ and $\hat{f}_j' = w' q_j'$ for some non-constant w'. If there is nothing to do according to this procedure one obtains a Gröbner basis. It has been proven by Buchberger [9] that such a procedure stops after a finite number of steps.

The Buchberger algorithm can take much computer time to construct a Gröbner basis, but once it has been constructed, one can use the reduction procedure which works generally much faster.

To conclude, the problem formulated in the beginning of this section can be solved by choosing an ordering and constructing the corresponding Gröbner basis using the Buch-

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berger algorithm. After that, one applies the reduction procedure modulo the constructed Gröbner basis to verify whether a given element belongs to the given ideal $\mathcal{I}$.

3. Reduction problem for Feynman integrals

Practically, one uses relations (1.3) of the following form:

$$\int \dots \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \dots \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \dots E_N^{a_N}} \right) = 0 \,. \tag{3.1}$$

Here E_r are denominators in (1.1), $k_1, \dots, k_h$ are loop momenta and $p_1 = k_1, \dots, p_h = k_h, p_{h+1} = q_1, \dots, p_{h+n} = q_n$, where $q_1, \dots, q_n$ are independent external momenta.

After the differentiation, resulting scalar products, $k_i \cdot k_j$ and $k_i \cdot q_j$ are expressed in terms of the factors in the denominator, by inverting (1.2), and one arrives at IBP relations which can be written as

$$\sum c_i F(a_1 + b_{i,1}, \dots, a_n + b_{i,n}) = 0 \,, \tag{3.2}$$

where $b_{i,j}$ are integer, c_i are polynomials in a_j, d, masses m_i and kinematic invariants, and $F(a_1, \dots, a_n)$ are Feynman integrals (1.1) of the given family. These relations can be written in terms of shift operators $\mathbf{i}^+$ and $\mathbf{i}^-$ which are defined as

$$\mathbf{i}^\pm \cdot F(a_1, a_2, \dots, a_n) = F(a_1, \dots, a_{i-1}, a_i \pm 1, a_{i+1}, \dots, a_n) \,.$$

At this point, we would like to turn from the 'physical' shift operators $\mathbf{i}^\pm$ to 'mathematical' shift operators. (We believe that the physical notation can be ambiguous: for example, it is not immediately clear whether the operators are applied to a function of the indices, or to some of its values.)

Let $\mathcal{K}$ be the field of rational functions of physical variables m_i, $q_i \cdot q_j$, d, and $\mathcal{A}$ be the algebra[2] over $\mathcal{K}$ generated by elements Y_i, Y_i^{-1} and A_i with the following relations:

$$\begin{aligned} Y_i Y_j &= Y_j Y_i, \quad A_i A_j = A_j A_i, \quad Y_i A_j = A_j Y_i + \delta_{i,j} Y_i, \\ Y_i^- Y_j^- &= Y_j^- Y_i^-, \quad Y_i^- Y_j = Y_j Y_i^-, \quad Y_i^- A_j = A_j Y_i^- - \delta_{i,j} Y_i, \quad Y_i^- Y_i = 1 \end{aligned} \tag{3.3}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. For convenience we will write $(Y_i^-)^k = Y_i^{-k}$. Let $\mathcal{F}$ be the field of functions of n integer arguments $a_1, a_2, \dots, a_n$. The algebra $\mathcal{A}$ acts on this field[3], where

$$\begin{aligned} (Y_i \cdot F)(a_1, a_2, \dots, a_n) &= F(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_n) \,, \\ (A_i \cdot F)(a_1, a_2, \dots, a_n) &= a_i F(a_1, a_2, \dots, a_n) \,. \end{aligned} \tag{3.4}$$

Let us turn back to the problem of calculating Feynman integrals. The left-hand sides of relations (3.2) can be represented as elements of the ring $\mathcal{A}$ applied to F; we will denote these elements by $f_1, \dots, f_n$. Now, for $F(a_1, \dots, a_n)$ defined by (1.1), we have

$$f_i \cdot F = 0 \text{ or } (f_i \cdot F)(a_1, \dots, a_n) = 0 \tag{3.5}$$

[2] An algebra over a field is a vector space over this field and a ring at the same time.

[3] (i) for any $a \in \mathcal{A}$ and $f \in \mathcal{F}$ we have an element $a \cdot f \in \mathcal{F}$; (ii) for any $a, b \in \mathcal{A}$ and $f, g \in \mathcal{F}$ we have $(a + b) \cdot (f + g) = a \cdot f + a \cdot g + b \cdot f + b \cdot g$; (iii) for any $a, b \in \mathcal{A}$ and $f \in \mathcal{F}$ we have $(ab) \cdot f = a \cdot (b \cdot f)$.

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for all i. Let us generate a (left) ideal $\mathcal{I}$ by the elements $f_1, \ldots, f_n$. We will call $\mathcal{I}$ the *ideal of the IBP relations*. Obviously,

$$f \cdot F = 0\,, \quad \text{or} \quad (f \cdot F)(a_1, \ldots, a_n) = 0 \quad \text{for any} \quad f \in \mathcal{I}\,. \tag{3.6}$$

Our goal is to express the value of F at an arbitrary point $(a_1, a_2, \ldots, a_n)$ in terms of the values of F in a few specially chosen points, i.e. master integrals. This problem can be solved similarly to the algebraic problem described in section 2. Consider, for example, the case, where all the indices a_i are positive. Then one has

$$F(a_1, a_2, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} \cdot F)(1, 1, \ldots, 1)\,. \tag{3.7}$$

The idea of the method is to reduce the monomial $Y_1^{a_1-1} \ldots Y_n^{a_n-1}$ modulo the ideal of IBP relations. Let us consider a trivial example of such a situation.

Example 1 . One-loop vacuum massive Feynman integrals

$$F(a) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a}\,. \tag{3.8}$$

Let us forget that these integrals can be evaluated explicitly, in terms of gamma functions. The IBP identity

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = 0\,, \tag{3.9}$$

leads to the relation

$$(d - 2a + 2)F(a - 1) - 2(a - 1)m^2 F(a) = 0\,. \tag{3.10}$$

We see that any Feynman integral $F(a)$ where $a > 1$ can be expressed recursively in terms of one integral $F(1) \equiv I_1$ which we therefore qualify as a master integral. (Observe that all the integrals with non-positive integer indices are integrals without scale and are naturally put to zero within dimensional regularization.)

Let us demonstrate how the reduction procedure can lead to the same result. (We realize that this way is more complicated in this simple situation. However, we will see later that its generalization provides simplifications and enables us to solve complicated problems.) The IBP relation (3.10) gives us one element $f = 2m^2 AY - (d - 2A) \in \mathcal{A}$ (the element $(f \cdot F)(a - 1)$ is the left-hand side of (3.10)). Set $\mathcal{I} = \mathcal{A}f$ (for any $g \in \mathcal{A}$ and $F \in \mathcal{F}$ we have $gf \cdot F$=0). We have

$$\begin{aligned} 2m^2(A + a - 2)Y^{a-1} &= (2m^2(A + a - 2)Y^{a-1} - Y^{a-2}f) + Y^{a-2}f \\ &= (2m^2(A + a - 2)Y^{a-1} - Y^{a-2}(2m^2 AY - (d - 2A))) + Y^{a-2}f \\ &= Y^{a-2}(d - 2A) + Y^{a-2}f = (d - 2A - 2a + 4)Y^{a-2} + Y^{a-2}f\,. \end{aligned} \tag{3.11}$$

The relation $2m^2(A + a - 2)Y^{a-1} = (d - 2A - 2a + 4)Y^{a-2} + X_1$, where $X_1 \in \mathcal{I}$, represents one step of the reduction procedure. If we stop the reduction at this point, we get

$$\begin{aligned} 2m^2(a - 1)F(a) &= (2m^2(a - 1) \cdot F)(a) = (2m^2(A + a - 2)Y^{a-1} \cdot F)(1) \\ &= ((d - 2A - 2a + 4)Y^{a-2}) \cdot F)(1) + (Y^{a-2}f \cdot F)(1) \\ &= (d - 2a + 2)F(a - 1) + ((A - 1)Y^{a-2} \cdot F)(1) \\ &= (d - 2a + 2)F(a - 1)\,, \end{aligned} \tag{3.12}$$

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i.e. the equation (3.10) we started from. But moving further with the reduction modulo $\mathcal{I}$ we obtain

$$P_1(A,a,m)Y^{a-1} = P_2(A,a,m) + X', \tag{3.13}$$

where P_1 and P_2 are polynomials obtained during the reduction and $X' \in \mathcal{I}$ (note that this algorithm is constructive and is realized as a computer code). Now we can apply this equation to F and take the value at 1:

$$\begin{aligned} P_1(1,a,m)F(a) &= (P_1(A,a,m)Y^{a-1}\cdot F)(1) \\ &= ((P_2(A,a,m)+X')\cdot F)(1) = P_2(1,a,m)F(1)\,. \end{aligned} \tag{3.14}$$

It is enough to notice that P_1 is a product of the leading coefficients (the formal definition in the non-commutative case will be given later) of f, Yf and so on. Thus $P_1(1,a,m)$ is the product of the leading coefficient $2m^2A$ of f with A replaced with all integers from 1 to $a-1$, hence non-zero. After dividing by this value we obtain the needed representation.

Thus it looks tempting to generalize the standard reduction procedure and reduce the monomial $Y_1^{a_1-1}\dots Y_n^{a_n-1}$ so that the resulting polynomial has a smaller degree in a certain sense. In this case we would represent $F(a_1,a_2,\dots,a_n)$ as a linear combination of $F(a_1',a_2',\dots,a_n')$ for 'smaller' a_i'.

This method works indeed, but first we need to introduce some notation. We will say that an element $X \in \mathcal{A}$ is written in the *proper* form if it is represented as

$$X = \sum c_j(A_1,\dots,A_n)\prod_i Y_i^{d_{i,j}}, \tag{3.15}$$

where c_j are polynomials and $d_{i,j}$ are integers. (So, all the operators A_i are placed on the left from the operators Y_i.) Obviously any element $X \in \mathcal{A}$ has a unique proper form. We will say that an element of $\mathcal{A}$ is a *monomial* if in its proper form only one coefficient function c_j is non-zero. We will say that the degree of a monomial $c(A_1,\dots,A_n)\prod_i Y_i^{d_i}$ is $\{d_1,\dots,d_n\}$. We will say that a monomial $c(A_1,\dots,A_n)\prod_i Y_i^{d_i}$ is *divisible* by a monomial $c'(A_1,\dots,A_n)\prod_i Y_i^{d_i'}$ if $d_i' \geq d_i$ for all i.

Let us define a subalgebra $\mathcal{A}^+ \subset \mathcal{A}$ generated as an algebra by Y_i and A_i (but not Y_i^{-1}) and set $\mathcal{I}^+ = \mathcal{I} \cap \mathcal{A}^+$. Obviously, $\mathcal{I}^+$ is an ideal in $\mathcal{A}^+$ and $\mathcal{A}\mathcal{I}^+ = \mathcal{I}$. In the same way as in the classical situation, we introduce the notion of an ordering, leading term, highest degree of an element of $\mathcal{A}^+$ and the reduction modulo an ideal. The only problem is that the leading coefficient of an element of $\mathcal{A}^+$ is now a polynomial function, so it does not generally have an inverse element. Thus the reduction procedures lead us to a relation

$$c_0(A_1,\dots,A_n)Y_1^{a_1-1}\dots Y_n^{a_n-1} = \sum_j c_j(A_1,\dots,A_n)\prod_i Y_i^{d_{i,j}} + X', \tag{3.16}$$

where $X' \in \mathcal{I}$, c_i are polynomials in A_j, and none of the monomials on the right-hand side of the relation is divisible by a leading monomial of an element of the basis $\mathcal{I}^+$. Applying this equality to F and taking the value at $(1,\dots,1)$ we derive

$$q_0F(a_1,\dots,a_n) = \sum_j q_jF(d_{1,j}+1,\dots,d_{n,j}+1)\,, \tag{3.17}$$

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where q_i do not depend on a_j. In the case where q_0 is non-zero we can divide by it and obtain the desired representation.

As in the classical situation, one says that a finite set $\{f_1, \ldots, f_k\}$ is a *Gröbner basis* of an ideal $\mathcal{I}$ if any element $f \in \mathcal{I}$ is reduced by this procedure to zero. The number of different degrees $\{d_1, \ldots, d_n\}$ arising on the right-hand side of (3.16) is minimal possible if we have a Gröbner basis. A Gröbner basis for the ideal of IBP relations can be constructed using the Buchberger algorithm (a generalization of the algorithm explained in section 2).

Let us now illustrate these points using a very simple

Example 2. The family of one-loop massless propagator integrals

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}((q-k)^2)^{a_2}} \,. \tag{3.18}$$

We have the boundary conditions, $F(a_1, a_2) = 0$ if $a_1 \leq 0$ or $a_2 \leq 0$, which correspond to putting to zero any integral without scale within dimensional regularization. As it is well known, this integral can be evaluated explicitly:

$$F(a_1, a_2) = i\pi^{d/2} \frac{(-1)^{a_1+a_2}\Gamma(a_1+a_2+\varepsilon-2)\Gamma(2-\varepsilon-a_1)\Gamma(2-\varepsilon-a_2)}{(-q^2)^{a_1+a_2-2+\varepsilon}\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\varepsilon)} \,, \tag{3.19}$$

but let us forget about this and consider the problem of the reduction to master integrals.

The two IBP identities

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \left(l \frac{1}{(k^2)^{a_1}((q-k)^2)^{a_2}} = 0 \right) \,, \tag{3.20}$$

with $l = k$ and $l = q$ give the following two IBP relations

$$d - 2a_1 - a_2 - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2) = 0 \,, \tag{3.21}$$

$$a_2 - a_1 - a_1 \mathbf{1}^+(q^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2) = 0 \,. \tag{3.22}$$

These relations are defined by the elements

$$f_1 = d - 2a_1 - a_2 - a_2 Y_2(Y_1^{-1} - q^2) \,, \tag{3.23}$$

$$f_2 = (a_2 - a_1) Y_1 Y_2 - a_1 Y_1(q^2 - Y_2^{-1}) - a_2 Y_2(Y_1^{-1} - q^2) \,. \tag{3.24}$$

which generate the ideal of IBP relations. Let us multiply (3.23) by Y_1 and (3.24) by $Y_1 Y_2$ to obtain a basis of $\mathcal{I}^+ \in \mathcal{A}^+$:

$$f_1' = (d - 2a_1 - a_2 - 2) Y_1 - a_2 Y_2(1 - q^2 Y_1) \,, \tag{3.25}$$

$$f_2' = a_2 - a_1 - (a_1 + 1) Y_1^2(q^2 Y_2 - 1) + (a_2 + 1) Y_2^2(q^2 Y_1 - 1) \,. \tag{3.26}$$

If we introduce an ordering for polynomials in the operators Y_i and define the corresponding reduction procedure modulo the operators f_1' and f_2' we shall obtain the possibility to represent any given monomial as

$$Y_1^{a_1-1} Y_2^{a_2-1} = r_1 f_1' + r_2 f_2' + \sum c_{ij} Y_1^{i-1} Y_2^{j-1} \,, \tag{3.27}$$

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where r_1 and r_2 are some elements of the ring $\mathcal{A}$. So, if we act by this relation on F and take the value at $(1,1)$ we shall obtain

$$F(a_1,a_2) = \sum_{i,j} c_{ij} F(i,j)\,. \tag{3.28}$$

We discover, however, that the set of integrals that appear on the right-hand side of these relations obtained for various a_1 and a_2 is infinite. The Buchberger algorithm described in section 2 leads, with the degree-lexicographic order, to the Gröbner basis consisting of the following two elements:

$$g_1 = 2a_1Y_1 - da_1Y_1 + 2a_1^2Y_1 - 2a_2Y_2 + da_2Y_2 - 2a_2^2Y_2 \tag{3.29}$$

$$\begin{aligned} g_2 = {} & 4a_1Y_1 - 4da_1Y_1 + d^2a_1Y_1 + 8a_1^2Y_1 - 4da_1^2Y_1 + 4a_1^3Y_1 + 2a_1a_2Y_1 \\ & -da_1a_2Y_1 + 2a_1^2a_2Y_1 + 2a_1a_2Y_2 - da_1a_2Y_2 + 2a_1^2a_2Y_2 - 4q^2a_2Y_2^2 \\ & +dq^2a_2Y_2^2 - 6q^2a_2^2Y_2^2 + dq^2a_2^2Y_2^2 - 2q^2a_2^3Y_2^2\,. \end{aligned} \tag{3.30}$$

Now, the reduction modulo these two elements provides only a finite number of integrals in the corresponding relations (3.28). In fact, the degree of g_1 is $(1,0)$ and the degree of g_2 is $(0,2)$, so we meet just the two integrals in this set, $F(1,1)$ and $F(1,2)$, and call them master integrals. For example, we have

$$F(2,3) = \frac{(d-8)(d-5)}{2q^2} F(1,2)\,. \tag{3.31}$$

However, we do not obtain a connection of $F(1,1)$ and $F(1,2)$, although we know, due to explicit solutions of the reduction procedure, that they are connected. This is a disturbing point. Of course, it is preferable to have only one master integral in this trivial example, so that we are going to develop an algorithm which reveals a minimal number of the master integrals at least in simple examples.

Starting from example 3, we will have at least one more complication: the variables a_i, generally, can be not only positive but also negative. (In the previous example, a_1 and a_2 were positive due to the boundary conditions.) Generally, we have to consider each variable a_i to be either positive or non-positive. Of course, for every family of Feynman integrals, there will be some boundary conditions. (In particular, if all the arguments a_i are non-positive any Feynman integral is zero.)

Thus, if we have a family of Feynman integrals, $F(a_1,\ldots,a_n)$, we are going to consider each variable a_i to satisfy $a_i > 0$ or $a_i \leq 0$. Consequently, we have to consider 2^n regions that we shall call *sectors* and label them by subsets $\nu \subseteq \{1,\ldots,n\}$. The corresponding sector σ_ν is defined as $\{(a_1,\ldots,a_n) : a_i > 0 \text{ if } i \in \nu\,,\ a_i \leq 0 \text{ if } i \notin \nu\}$.

In the sector where all a_i are positive, we considered the ring $\mathcal{A}^+ \subset \mathcal{A}$ and the operators Y_i as basic operators. (See the previous example.) Quite similarly, in a given sector σ_ν it is natural to consider the subalgebra $\mathcal{A}^\nu \subset \mathcal{A}$ generated by the operators A_i and the operators Y_i for $i \in \nu$ and Y_i^{-1} for other i. Within this definition we have $\mathcal{A}^{\{1,\ldots,n\}} = \mathcal{A}^+$.

Thus the first idea is to construct a Gröbner basis for each of the 2^n sectors, or at least for all non-trivial sectors. (We call a sector *trivial* if all the given Feynman integrals

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are identically zero in it due to boundary conditions.) This approach however faces many problems:

1. Each of the non-trivial sectors will give us at least one point where we have to evaluate F;

2. The number of points where we have to evaluate F in a given sector is generally greater than the real number of master integrals (In the last example there is only one master integral but we obtain $F(1,1)$ and $F(1,2)$ after the reduction);

3. Even if one has constructed all the needed bases, the reduction may fail in cases where the coefficient q_0 in eq. (3.17) is zero. This problem arises because all leading coefficients are polynomials in A_i, so that they can be equal to zero at certain points;

4. Although the method leads us theoretically to constructing a Gröbner basis, all known practical implementations fail to work even already in four-dimensional examples.

Therefore this specialization of the Buchberger algorithm turns out to be completely impractical in sufficiently complicated examples. Our algorithm is a certain modification of the Buchberger algorithm. To characterize it we need to introduce some notation.

Let $\mathcal{A}^{(\nu)} = \oplus_{\nu' \subseteq \nu} \mathcal{A}^{\nu'} \subset \mathcal{A}$. First of all let us define a *sector-reduction*, or s-reduction of an element $f \in \mathcal{A}^{(\nu)}$ modulo a basis of the ideal $\mathcal{I}^\nu = \mathcal{I} \cap \mathcal{A}^\nu$. Take the proper form of f and let f^ν be the sum of the terms in this decomposition that lie in $\mathcal{A}^\nu$. If f^ν is equal to zero the s-reduction stops. Otherwise we look for a monomial $g \in \mathcal{A}^{(\nu)}$ and a coefficient $c \in \mathcal{K}$ such that the degrees of f and gf_i for some element of the bases f_i coincide, that $(cf - gf_i)^\nu$ is zero or its degree is smaller and that the value of c at the point $(a_1, \ldots, a_n)$ is non-zero, where $a_i = 1$ if $i \in \nu$ and 0 otherwise. The procedure is repeated while possible.

A *sector* basis, or, an s-basis for a sector σ_ν is a basis of the ideal I^ν such that the number of possible degrees f^ν, where f is the result of the s-reduction, is finite. Such a basis provides the possibility of a reduction to master integrals *and* integrals whose indices lie in *lower* sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$.

To prove that an s-basis always exists is an open problem, but in all our examples they do exist, and it turns out that in all known examples where one can construct a Gröbner basis it is an s-basis as well, although this does not follow from the definition.

If $\nu = \emptyset$ then an s-basis is a Gröbner basis but generally it is not. Since the sector $\sigma_\emptyset$ is trivial, we do not have to construct a single Gröbner bases. Still, it is most complicated to construct s-bases for minimal sectors (a sector σ_ν is said to be *minimal* if it is non-trivial but all lower sectors are trivial).

Having constructed s-bases for all non-trivial sectors we have an algorithm to evaluate F at any point. Indeed, we choose a sector containing the point we need, run the s-reduction algorithm for this sector, expressing F in terms of some master integrals and values for lowers sectors, then repeat the procedure for all those sectors. Eventually we reduce F to the master integrals.

The Buchberger algorithm leads us to constructing a Gröbner basis that is hopefully an s-basis, but this has no use for us since this does not simplify anything. The second

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important point is that the Buchberger algorithm can be terminated when the Gröbner basis is not yet constructed but the 'current' basis already provides us the s-reduction, so that it is an s-basis (we have criteria that show whether a basis is an s-basis).

Let us illustrate how this idea works on the same example. The initial basis turns out to be an s-basis. First, let us observe that the degree of f_1' is $(1,1)$ and the leading coefficient is q^2a_2, i.e. is a non-zero function in the positive sector, hence we are capable of making reduction steps if the highest degree of an element being reduced is different from $(l,0)$ and $(0,l)$. Now let f be a polynomial whose highest degree is $(l,0)$ and the leading coefficient is c. Then

$$f' = q^2(l-1+a_1)f + cY_1^{l-2}Y_2^{-1}f_2' \tag{3.32}$$

is an element of $\mathcal{A}$. Let us take the proper form of f (implying that the numbers $d_{i,j}$ can be now negative) and calculate its highest degree without paying attention to the terms with negative $d_{i,j}$. Obviously it is smaller than the degree of f. Now if we take the value of f' at $(1,1)$ we will have the elements like $F(j,0)$ among others. But the boundary conditions state that they are equal to zero, so that we have nothing to worry about. Now we can move further and reduce f' as we did before. Finally we get

$$f = X + c(A_1, A_2) + Y \tag{3.33}$$

where $X \in \mathcal{I}$, c is a rational function and Y is an element of $\mathcal{A}$ such that in its proper form there are no degrees d_1, d_2 with $d_1 \geq 0$ and $d_2 \geq 0$. Taking the value at $(1,1)$ we obtain the reduction of any value of F to the value of the master integral $F(1,1)$. (See also [13].)

4. Examples

Let us consider a modification of Example 2; now we have a non-zero mass $m_1 = m$:

Example 3. Propagator integrals with the masses m and 0,

$$F(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(k^2-m^2)^{a_1}[(q-k)^2]^{a_2}} \,. \tag{4.1}$$

The integrals are zero if $a_1 \leq 0$. The corresponding IBP relations generate the following elements:

$$\begin{aligned} f_1 &= d - 2a_1 - a_2 - 2m^2a_1Y_1 - m^2a_2Y_2 + q^2a_2Y_2 - a_2Y_2Y_1^{-1} \\ f_2 &= a_2 - a_1 - m^2a_1Y_1 - q^2a_1Y_1 - m^2a_2Y_2 + q^2a_2Y_2 - a_2Y_2Y_1^{-1} + a_1Y_1Y_2^{-1} \,. \end{aligned}$$

We have to consider two sectors, $\sigma_{\{1,2\}}$ and $\sigma_{\{1\}}$.

Using the lexicographical ordering, we obtain, for the sector $\sigma_{\{1,2\}}$, the s-basis consisting of two elements:

$$\begin{aligned} g_{11} &= Y_1^2 + a_1Y_1^2 + 3Y_1Y_2 - dY_1Y_2 + a_1Y_1Y_2 + 2a_2Y_1Y_2 + m^2Y_1^2Y_2 \\ &\quad -q^2Y_1^2Y_2 + m^2a_1Y_1^2Y_2 - q^2a_1Y_1^2Y_2 \,, \\ g_{12} &= -3Y_1Y_2 + dY_1Y_2 - 2a_1Y_1Y_2 - a_2Y_1Y_2 - 2m^2Y_1^2Y_2 - 2m^2a_1Y_1^2Y_2 - Y_2^2 \\ &\quad -a_2Y_2^2 - m^2Y_1Y_2^2 + q^2Y_1Y_2^2 - m^2a_2Y_1Y_2^2 + q^2a_2Y_1Y_2^2 \,. \end{aligned}$$

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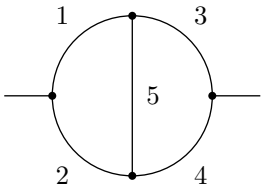

Figure 1: Two-loop propagator diagram.

For the sector $\sigma_{\{1\}}$, we obtain the following s-basis:

$$\begin{aligned}
g_{21} &= 1 - a_2 + m^2 Y_1 - q^2 Y_1 - m^2 a_2 Y_1 + q^2 a_2 Y_1 - Y_1 Y_2^{-1} + d Y_1 Y_2^{-1} - 2a_1 Y_1 Y_2^{-1} \\
&\quad - a_2 Y_1 Y_2^{-1} - 2m^2 Y_1^2 Y_2^{-1} - 2m^2 a_1 Y_1^2 Y_2^{-1} \,, \\
g_{22} &= -2m^2 + 2m^2 a_2 - 2m^4 Y_1 + 2m^2 q^2 Y_1 + 2m^4 a_2 Y_1 - 2m^2 q^2 a_2 Y_1 - 2Y_2^{-1} + a_2 Y_2^{-1} \\
&\quad + 2m^2 Y_1 Y_2^{-1} + 2q^2 Y_1 Y_2^{-1} + 2m^2 a_1 Y_1 Y_2^{-1} - m^2 a_2 Y_1 Y_2^{-1} - q^2 a_2 Y_1 Y_2^{-1} \\
&\quad + 2m^4 Y_1^2 Y_2^{-1} + 2m^2 q^2 Y_1^2 Y_2^{-1} + 2m^4 a_1 Y_1^2 Y_2^{-1} + 2m^2 q^2 a_1 Y_1^2 Y_2^{-1} \\
&\quad - d Y_1 Y_2^{-2} + 2a_1 Y_1 Y_2^{-2} + a_2 Y_1 Y_2^{-2} \,.
\end{aligned}$$

The reduction based on the two constructed s-sectors reveals two master integrals, $F(1,1)$ and $F(1,0)$, in accordance with results obtained by other ways. (See, e.g., Chapters 5 and 6 of [15] and [13].)

Example 4. Two-loop massless propagator diagram of figure 1.

The corresponding family of Feynman integrals is

$$F(a_1,a_2,a_3,a_4,a_5) = \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2)^{a_1}[(q-k)^2]^{a_2}(l^2)^{a_3}[(q-l)^2]^{a_4}[(k-l)^2]^{a_5}} \,. \tag{4.2}$$

There are boundary conditions which correspond to setting to zero integrals without scale: $F(a_1,a_2,a_3,a_4,a_5) = 0$, if $a_i, a_5 \leq 0$ for $i = 1,\ldots,4$, or $a_1, a_2 \leq 0$, or $a_3, a_4 \leq 0$, or $a_1, a_3 \leq 0$, or $a_2, a_4 \leq 0$. The integrals are symmetrical:

$$F(a_1,a_2,a_3,a_4,a_5) = F(a_2,a_1,a_4,a_3,a_5) = F(a_3,a_4,a_1,a_2,a_5) \,.$$

The corresponding IBP relations generate the following elements:

$$\begin{aligned}
f_1 &= (d - 2a_1 - a_2 - a_5) + a_2 Y_2 (q^2 - Y_1^{-1}) - a_5 Y_5 (Y_1^{-1} - Y_3^{-1}) \,, \\
f_2 &= (d - a_2 - 2a_3 - a_5) + a_4 Y_4 (q^2 - Y_3^{-1}) - a_5 Y_5 (Y_3^{-1} - Y_1^{-1}) \,, \\
f_3 &= (d - a_1 - a_2 - 2a_5) + a_1 Y_1 (Y_3^{-1} - Y_5^{-1}) + a_2 Y_2 (Y_4^{-1} - Y_5^{-1}) \,, \\
f_4 &= (d - a_3 - a_4 - 2a_5) + a_3 Y_3 (Y_1^{-1} - Y_5^{-1}) + a_4 Y_4 (Y_2^{-1} - Y_5^{-1}) \,, \\
f_5 &= (d - a_1 - 2a_2 - a_5) + a_1 Y_1 (q^2 - Y_2^{-1}) - a_5 Y_5 (Y_2^{-1} - Y_4^{-1}) \,, \\
f_6 &= (d - a_3 - 2a_4 - a_5) + a_3 Y_3 (q^2 - Y_4^{-1}) - a_5 Y_5 (Y_4^{-1} - Y_2^{-1}) \,.
\end{aligned}$$

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As is well known, any integral of this class can be reduced, due to IBP relations, in a very simple way, to integrals where at least one of the indices is non-positive. Such integrals can be evaluated recursively in terms of gamma functions using the one-loop integration formula (3.19). Let us point out that physicists often stop the reduction whenever they arrive at integrals expressed in terms of gamma functions. Imagine, however, that we want to know the whole solution of the reduction procedure, i.e. a reduction to a minimal number of the master integrals. Then this example turns out be not so trivial and provides a good possibility to test our algorithms.

This can be done in various ways. In our approach, we apply our algorithm to construct s-bases corresponding to the sectors $\sigma_{\{1,2,3,4,5\}}$, $\sigma_{\{2,3,4,5\}}$ as well as three more (symmetrical) sectors, the minimal sector $\sigma_{\{1,2,3,4\}}$, the minimal sector $\sigma_{\{2,3,5\}}$ as well as one more (symmetrical) sector. In the first three cases, we used the degree-lexicographical ordering, and in the last case, some special ordering.

For example, let us present the s-basis associated with $\sigma_{\{1234\}}$:

$$
\begin{aligned}
g_1 = {} & 2Y_1Y_2Y_3Y_5^{-1} - a_5Y_1Y_2Y_3Y_5^{-1} - 2Y_1Y_2Y_4Y_5^{-1} + a_5Y_1Y_2Y_4Y_5^{-1} - 2Y_1Y_3Y_4Y_5^{-1} \\
& + a_5Y_1Y_3Y_4Y_5^{-1} + 2Y_2Y_3Y_4Y_5^{-1} - a_5Y_2Y_3Y_4Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4Y_5^{-1} \\
& - q^2a_5Y_1Y_2Y_3Y_4Y_5^{-1} + 2q^2Y_1Y_2Y_4^2Y_5^{-1} - q^2a_5Y_1Y_2Y_4^2Y_5^{-1} - 2q^2Y_1Y_3Y_4^2Y_5^{-1} \\
& + q^2a_5Y_1Y_3Y_4^2Y_5^{-1} - 2q^2Y_2Y_3Y_4^2Y_5^{-1} + q^2a_5Y_2Y_3Y_4^2Y_5^{-1} - (q^2)^2Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& + d(q^2)^2Y_1Y_2Y_3Y_4^2Y_5^{-1} - (q^2)^2a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} - (q^2)^2a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& - 2(q^2)^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} + (q^2)^2Y_2Y_3^2Y_4^2Y_5^{-1} + (q^2)^2a_3Y_2Y_3^2Y_4^2Y_5^{-1} \\
& + 2(q^2)^2Y_1Y_3Y_4^3Y_5^{-1} + (q^2)^2a_4Y_1Y_3Y_4^3Y_5^{-1} - Y_1Y_2Y_3^2Y_5^{-2} - a_3Y_1Y_2Y_3^2Y_5^{-2} \\
& + a_3Y_1Y_2Y_3Y_4Y_5^{-2} - a_4Y_1Y_2Y_3Y_4Y_5^{-2} + Y_1Y_2Y_4^2Y_5^{-2} + a_4Y_1Y_2Y_4^2Y_5^{-2} \\
& - 6q^2Y_1Y_2Y_3Y_4^2Y_5^{-2} + 2dq^2Y_1Y_2Y_3Y_4^2Y_5^{-2} - 3q^2a_3Y_1Y_2Y_3Y_4^2Y_5^{-2} \\
& - 4q^2a_4Y_1Y_2Y_3Y_4^2Y_5^{-2} - 2q^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-2} - 2q^2Y_1Y_2Y_4^3Y_5^{-2} \\
& - q^2a_4Y_1Y_2Y_4^3Y_5^{-2} \,, \\
g_2 = {} & Y_1Y_2Y_4^2 - a_5Y_1Y_2Y_4^2 - Y_2Y_3Y_4^2 + a_5Y_2Y_3Y_4^2 - 3Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& + dY_1Y_2Y_3Y_4^2Y_5^{-1} - 2a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& - 2Y_1Y_2Y_4^3Y_5^{-1} - a_4Y_1Y_2Y_4^3Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^3Y_5^{-1} + q^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-1} \,, \\
g_3 = {} & -Y_1Y_2Y_3Y_4 + a_5Y_1Y_2Y_3Y_4 + Y_1Y_2Y_4^2 - a_5Y_1Y_2Y_4^2 + Y_1Y_3Y_4^2 - a_5Y_1Y_3Y_4^2 \\
& - Y_2Y_3Y_4^2 + a_5Y_2Y_3Y_4^2 + Y_1Y_2Y_3^2Y_4Y_5^{-1} + a_3Y_1Y_2Y_3^2Y_4Y_5^{-1} + Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& - a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} + a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} - q^2Y_1Y_2Y_3^2Y_4^2Y_5^{-1} - q^2a_3Y_1Y_2Y_3^2Y_4^2Y_5^{-1} \\
& - 2Y_1Y_2Y_4^3Y_5^{-1} - a_4Y_1Y_2Y_4^3Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^3Y_5^{-1} + q^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-1} \,, \\
g_4 = {} & -Y_1Y_2Y_4^2 + a_5Y_1Y_2Y_4^2 + Y_2Y_3Y_4^2 - a_5Y_2Y_3Y_4^2 - 2Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& + dY_1Y_2Y_3Y_4^2Y_5^{-1} - 2a_1Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_2Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& - Y_2^2Y_3Y_4^2Y_5^{-1} - a_2Y_2^2Y_3Y_4^2Y_5^{-1} + q^2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} + q^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} \,, \\
g_5 = {} & Y_1Y_2Y_3Y_4 - a_5Y_1Y_2Y_3Y_4 - Y_1Y_2Y_4^2 + a_5Y_1Y_2Y_4^2 - Y_1Y_3Y_4^2 + a_5Y_1Y_3Y_4^2 \\
& + Y_2Y_3Y_4^2 - a_5Y_2Y_3Y_4^2 + Y_1^2Y_3Y_4^2Y_5^{-1} + a_1Y_1^2Y_3Y_4^2Y_5^{-1} - a_1Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& + a_2Y_1Y_2Y_3Y_4^2Y_5^{-1} - q^2Y_1^2Y_2Y_3Y_4^2Y_5^{-1} - q^2a_1Y_1^2Y_2Y_3Y_4^2Y_5^{-1} - Y_2^2Y_3Y_4^2Y_5^{-1}
\end{aligned}
$$

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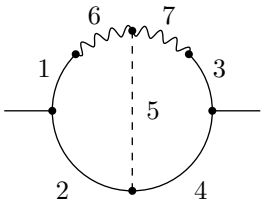

Figure 2: Feynman diagram contributing to the three-loop static quark potential. A wavy line denotes a propagator for the static source and the dotted line denotes the scalar propagator with the index shifted by ε.

$$
\begin{aligned}
& -a_2Y_2^2Y_3Y_4^2Y_5^{-1} + q^2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} + q^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} \,, \\
g_6 = {} & -q^2Y_1Y_2Y_4^2 + q^2a_5Y_1Y_2Y_4^2 + q^2Y_2Y_3Y_4^2 - q^2a_5Y_2Y_3Y_4^2 + 2Y_1Y_2Y_3Y_4Y_5^{-1} \\
& -a_5Y_1Y_2Y_3Y_4Y_5^{-1} - 2Y_1Y_2Y_4^2Y_5^{-1} + a_5Y_1Y_2Y_4^2Y_5^{-1} - 2Y_1Y_3Y_4^2Y_5^{-1} + a_5Y_1Y_3Y_4^2Y_5^{-1} \\
& +2Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_2Y_3Y_4^2Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^2Y_5^{-1} + q^2a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& -q^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} + q^2Y_2Y_3^2Y_4^2Y_5^{-1} + q^2a_3Y_2Y_3^2Y_4^2Y_5^{-1} + 2q^2Y_1Y_2Y_4^3Y_5^{-1} \\
& +q^2a_4Y_1Y_2Y_4^3Y_5^{-1} + 2q^2Y_1Y_3Y_4^3Y_5^{-1} + q^2a_4Y_1Y_3Y_4^3Y_5^{-1} - 2(q^2)^2Y_1Y_2Y_3Y_4^3Y_5^{-1} \\
& -(q^2)^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-1} - Y_1Y_2Y_3^2Y_4Y_5^{-2} - a_3Y_1Y_2Y_3^2Y_4Y_5^{-2} - Y_1Y_2Y_3Y_4^2Y_5^{-2} \\
& +a_3Y_1Y_2Y_3Y_4^2Y_5^{-2} - a_4Y_1Y_2Y_3Y_4^2Y_5^{-2} + 2Y_1Y_2Y_4^3Y_5^{-2} + a_4Y_1Y_2Y_4^3Y_5^{-2} \\
& -4q^2Y_1Y_2Y_3Y_4^3Y_5^{-2} - 2q^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-2} \,, \\
g_7 = {} & q^2Y_1Y_2Y_4^2 - q^2a_5Y_1Y_2Y_4^2 - q^2Y_2Y_3Y_4^2 + q^2a_5Y_2Y_3Y_4^2 - 2Y_1Y_2Y_3Y_4Y_5^{-1} \\
& +a_5Y_1Y_2Y_3Y_4Y_5^{-1} + q^2Y_1Y_2^2Y_3Y_4Y_5^{-1} + q^2a_2Y_1Y_2^2Y_3Y_4Y_5^{-1} + 2Y_1Y_2Y_4^2Y_5^{-1} \\
& -a_5Y_1Y_2Y_4^2Y_5^{-1} + q^2Y_1^2Y_2Y_4^2Y_5^{-1} + q^2a_1Y_1^2Y_2Y_4^2Y_5^{-1} + 2Y_1Y_3Y_4^2Y_5^{-1} \\
& -a_5Y_1Y_3Y_4^2Y_5^{-1} - 2Y_2Y_3Y_4^2Y_5^{-1} + a_5Y_2Y_3Y_4^2Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& +q^2a_1Y_1Y_2Y_3Y_4^2Y_5^{-1} - q^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} + q^2Y_2^2Y_3Y_4^2Y_5^{-1} + q^2a_2Y_2^2Y_3Y_4^2Y_5^{-1} \\
& -(q^2)^2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} - (q^2)^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} - Y_1^2Y_3Y_4^2Y_5^{-2} - a_1Y_1^2Y_3Y_4^2Y_5^{-2} \\
& +a_1Y_1Y_2Y_3Y_4^2Y_5^{-2} - a_2Y_1Y_2Y_3Y_4^2Y_5^{-2} + Y_2^2Y_3Y_4^2Y_5^{-2} \\
& +a_2Y_2^2Y_3Y_4^2Y_5^{-2} - 2q^2Y_1Y_2^2Y_3Y_4^2Y_5^{-2} - 2q^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-2} \,.
\end{aligned}
$$

The reduction based on the constructed s-sectors reveals three master integrals, $F(1,1,1,1,0)$, $F(0,1,1,0,1)$ and $F(1,0,0,1,1)$ (the last two of them are equal because of the symmetry), in accordance with results obtained by other ways. (See, e.g., Chapters 5 and 6 of [15].)

Our last example is

Example 5. Two-loop Feynman integrals for the heavy quark static potential corresponding to figure 2 with $v \cdot q = 0$.

We shall consider diagrams contributing to the three-loop static potential corresponding to figure 2. They are obtained from the corresponding two-loop diagrams by inserting a one-loop diagram into the central line. Indeed, the integration over the loop-momentum of the insertion can be performed explicitly, by means of (3.19), and one obtains, up to

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a factor expressed in terms of gamma functions, Feynman integrals of figure 2, where the index of the central line[4] is $a_5+\varepsilon \equiv a_5+(4-d)/2$ with integer a_5. So, we arrive at the following family of integrals:[5]

$$F(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = \int\int \frac{\mathrm{d}^d k \mathrm{d}^d l}{(-k^2)^{a_1}(-l^2)^{a_2}[-(k-q)^2]^{a_3}[-(l-q)^2]^{a_4}} \times \frac{1}{[-(k-l)^2]^{a_5+\varepsilon}(-v\cdot k)^{a_6}(-v\cdot l)^{a_7}}\,. \tag{4.3}$$

We have turned to the $-k^2$ dependence of the propagators because this choice is more natural when at least one index, $a_5+\varepsilon$ is not integer. The integrals are symmetrical:

$$F(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = F(a_2,a_1,a_4,a_3,a_5,a_6,a_7) = F(a_3,a_4,a_1,a_2,a_5,a_7,a_6)\,.$$

They are equal to zero, if $a_1,a_3\le 0$, or $a_2,a_4\le 0$, or $a_1,a_2,a_6\le 0$, or $a_3,a_4,a_7\le 0$.

The IBP relations generate the following elements in the case where a_5 is not shifted by ε (i.e. the case of the diagrams relevant to the two-loop static quark potential considered in [14, 8])

$$\begin{aligned}
f_1 &= (d-2a_1-a_2-a_5-a_6) - a_2Y_2(q^2+Y_1^{-1}) - a_5Y_5(Y_1^{-1}-Y_3^{-1})\,,\\
f_2 &= (d-a_2-2a_3-a_5-a_7) - a_4Y_4(q^2+Y_3^{-1}) - a_5Y_5(Y_3^{-1}-Y_1^{-1})\,,\\
f_3 &= (d-a_1-a_2-2a_5-a_6) + a_1Y_1(Y_3^{-1}-Y_5^{-1}) + a_2Y_2(Y_4^{-1}-Y_5^{-1}) + a_6Y_6Y_7^{-1}\,,\\
f_4 &= (d-a_3-a_4-2a_5-a_7) + a_3Y_3(Y_1^{-1}-Y_5^{-1}) + a_4Y_4(Y_2^{-1}-Y_5^{-1}) + a_7Y_6^{-1}Y_7\,,\\
f_5 &= (d-a_1-2a_2-a_5-a_6) - a_1Y_1(q^2+Y_2^{-1}) - a_5Y_5(Y_2^{-1}-Y_4^{-1})\,,\\
f_6 &= (d-a_3-2a_4-a_5-a_7) - a_3Y_3(q^2+Y_4^{-1}) - a_5Y_5(Y_4^{-1}-Y_2^{-1})\,,\\
f_7 &= 2a_1Y_1Y_6^{-1} + 2a_2Y_2Y_6^{-1} + a_5Y_5(Y_6^{-1}-Y_7^{-1}) - v^2a_6Y_6\,,\\
f_8 &= 2a_3Y_3Y_7^{-1} + 2a_4Y_4Y_7^{-1} - a_5Y_5(Y_6^{-1}-Y_7^{-1}) - v^2a_7Y_7\,.
\end{aligned}$$

So, the IBP elements we need are obtained from these by replacing a_5 with $a_5+\varepsilon$.

Our algorithm works successfully in this example and gives us a family of s-bases which provide the possibility of a reduction to master integrals. The elements of the bases are rather lengthy, typically, with hundreds of terms, so that we do not present them in this short paper. These s-bases correspond to the following sectors: $\sigma_{\{1,2,3,4,5,6,7\}}$, $\sigma_{\{2,3,4,5,6,7\}}$, $\sigma_{\{1,2,3,4,5,7\}}$, $\sigma_{\{3,4,5,6,7\}}$, $\sigma_{\{2,3,5,6,7\}}$, $\sigma_{\{2,3,4,5,7\}}$, $\sigma_{\{2,3,4,5,6\}}$, $\sigma_{\{1,2,3,4,5\}}$, $\sigma_{\{2,3,4,5\}}$, $\sigma_{\{2,3,5,6\}}$ and other sectors obtained by the symmetry transformations.

We obtain the master integrals: $I_1 = F(1,1,1,1,0,1,1), I_{21} = F(1,1,1,1,0,0,1)$, $I_{22} = F(1,1,1,1,0,1,0), I_3 = F(1,1,1,1,0,0,0)$. We have $I_{21} = I_{12} = I_2$ because of the symmetry. We also obtain $I_{51} = F(1,0,0,1,1,1,1), I_{71} = F(1,0,0,1,1,0,1), I_{81} = F(1,0,0,1,1,1,0), I_{41} = F(1,0,0,1,1,0,0)$. We have $I_{71} = I_{81} = I_7$ because of the symmetry. Moreover, we have other copies, $I_{52}, I_{72}, I_{82}, I_{42}$, of this last family of the master

[4] A more general case, with $a_5 \to a_5 + r\varepsilon$ and integer r, relevant to r-loop massless insertions can be considered on the same footing.

[5] These integrals with the integer index of the central line contribute to the two-loop static quark potential and were calculated in [14, 8].

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integrals which are obtained by the symmetry transformation $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$. We also obtain $I_{61} = F(0,0,1,1,1,1,0), \bar{I}_{61} = F(0,0,1,1,1,2,0)$ as well as the corresponding symmetrical family.

To calculate the master integrals one can use the threefold Mellin-Barnes representation

$$
\begin{aligned}
F(a_1,a_2,a_3,a_4,a_5,a_6,a_7) &= \frac{\left(i\pi^{d/2}\right)^2 2^{a_7-1}(v^2)^{-a_{67}/2}}{\prod_{l=3,4,5,7}\Gamma(a_l)\Gamma(4-a_{3457}-2\varepsilon)(Q^2)^{a_{12345}-4+2\varepsilon+a_{67}/2}} \\
&\times \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty}\dots\int_{-i\infty}^{+i\infty} \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \frac{\Gamma(a_{12345}+a_{67}/2+2\varepsilon-4+z_3)}{\Gamma(a_{345}+a_{67}/2+\varepsilon-3/2+z_1+z_2+z_3)} \\
&\times \frac{\Gamma(a_3+z_1+z_3)\Gamma(a_4+z_2+z_3)\Gamma(a_{345}+a_7/2+\varepsilon-2+z_1+z_2+z_3)}{\Gamma(a_1-z_1)\Gamma(a_2-z_2)\Gamma(8-a_{1267}-2a_{345}-4\varepsilon-z_1-z_2-2z_3)} \\
&\times \Gamma(a_{345}+a_7/2+\varepsilon-3/2+z_1+z_2+z_3)\Gamma(4-2a_{34}-a_{57}-2\varepsilon-z_1-z_2-2z_3) \\
&\times \Gamma(4-a_{1345}-a_{67}/2-2\varepsilon-z_2-z_3)\Gamma(2-a_{345}-\varepsilon-z_1-z_2-z_3) \\
&\times \Gamma(4-a_{2345}-a_{67}/2-2\varepsilon-z_1-z_3)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\,, \qquad (4.4)
\end{aligned}
$$

where $a_{12345} = a_1+a_2+a_3+a_4+a_5$ etc. The integrations over the variables z_i go from $-\infty$ to $+\infty$ in the complex plane. The contours are chosen in the standard way: the poles of gamma functions with the $-z_i$ dependence are to the right of the contour and the poles with the $+z_i$ dependence are to the left of it. This representation can be derived by applying Feynman parameterization to the subloop integral over l, introducing then three MB integrations and, finally, integrating over k. (See Chapter 4 of [15] for details of this method, with multiple examples.)

We obtain the following results for the master integrals:

$$
\begin{aligned}
I_1 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{Q^{4+6\varepsilon}v^2}\left[-\frac{8\pi^2}{9\varepsilon}-\frac{16\pi^2}{9}+\frac{40\zeta(3)}{3}+O(\varepsilon)\right]\,, \\
I_2 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{Q^{3+6\varepsilon}v}\left[\frac{\pi^4}{3}+O(\varepsilon)\right]\,, \\
I_3 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{Q^{2+6\varepsilon}}\left[6\zeta(3)+\left(\frac{\pi^4}{10}+12\zeta(3)\right)\varepsilon+O(\varepsilon^2)\right]\,, \\
I_4 &= \left(i\pi^{d/2}\right)^2 Q^{2-6\varepsilon}\frac{\Gamma(1-2\varepsilon)\,\Gamma(1-\varepsilon)^2\,\Gamma(3\varepsilon-1)}{\Gamma(3-4\varepsilon)\,\Gamma(1+\varepsilon)}\,, \\
I_5 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{Q^{6\varepsilon-2}v^2}\left[\frac{4\pi^2}{9\varepsilon}+\frac{32\pi^2}{9}-\frac{8\zeta(3)}{3}+O(\varepsilon)\right]\,, \\
I_6 &= \left(i\pi^{d/2}\right)^2 Q^{2-6\varepsilon}\frac{4^{1-2\varepsilon}\sqrt{\pi}\Gamma(3/2-3\varepsilon)^2\,\Gamma(1-2\varepsilon)\,\Gamma(3\varepsilon-1/2)\,\Gamma(4\varepsilon-1)}{\Gamma(3-6\varepsilon)\,\Gamma(2\varepsilon)\,\Gamma(1+\varepsilon)\,v}\,, \\
\bar{I}_6 &= \left(i\pi^{d/2}\right)^2 Q^{2-6\varepsilon}\frac{4^{1-2\varepsilon}\sqrt{\pi}\Gamma(1-3\varepsilon)^2\,\Gamma(1-2\varepsilon)\,\Gamma(3\varepsilon)\,\Gamma(4\varepsilon)}{\Gamma(2-6\varepsilon)\,\Gamma(1+\varepsilon)\,\Gamma(1/2+2\varepsilon)\,v^2}\,, \\
I_7 &= \left(i\pi^{d/2}\right)^2 \frac{\sqrt{\pi}\Gamma(3/2-3\varepsilon)\,\Gamma(1-2\varepsilon)\,\Gamma(1/2-\varepsilon)\,\Gamma(1-\varepsilon)\,\Gamma(3\varepsilon-1/2)}{Q^{6\varepsilon-1}v\Gamma(2-4\varepsilon)\,\Gamma(2-3\varepsilon)\,\Gamma(1+\varepsilon)}\,, \\
I_8 &= I_7\,,
\end{aligned}
$$

where $Q = \sqrt{-q^2}$ and $v = \sqrt{v^2}$.

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Observe that some of the integrals are expressed explicitly in terms of gamma functions for general values of ε while results for some other integrals are presented in expansion in a Laurent series in ε. The depth of this expansion can be made greater whenever necessary.

For example, we obtain the following reductions to master integrals by our algorithm:

$$\begin{aligned} F(1,1,1,1,1,1,-1) &= -\frac{2Q^2v^2}{(3d-10)}\bar{I}_2 - 3I_3 - \frac{8(d-3)(2d-7)(11d-46)}{(d-4)^2(3d-14)Q^4}I_4 \\ &\quad + \frac{4(3d-11)(7d-30)v^2}{(d-4)(3d-14)(3d-10)Q^2}\bar{I}_6\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} F(2,1,1,1,1,1,1) &= -\frac{3d-14}{2Q^2}I_1 \\ &\quad - \frac{4(d-3)(d-2)(2d-7)(3d-10)(9d-40)}{(d-5)(d-4)(2d-11)(3d-16)(3d-14)Q^8v^2}I_4 \\ &\quad - \frac{3(d-4)(4d-17)(4d-15)}{(2(d-5)(2d-11)Q^6}I_5 - \frac{16(3d-13)(3d-11)}{(2d-11)(3d-16)(3d-14)Q^6}\bar{I}_6\,, \end{aligned} \tag{4.6}$$

which can be checked straightforwardly, by evaluating these integrals, in expansion in ε, using the MB representation (4.4).

5. Conclusion

We have developed an algorithm which is a generalization of the Buchberger algorithm to the reduction problem for Feynman integrals and modified it in such a way that it works at the level of modern calculations. We have described the main features of the algorithm. For the examples considered, it works rather fast — these are seconds of CPU time for Example 4 and minutes for Example 5, both for constructing s-bases and reduction to master integrals. In fact, it has turned out that our algorithm works successfully even at a higher level, in a reduction problem with nine indices [16]. Still to perform more sophisticated calculations, further modifications and optimizations are needed. One of possible ways to improve the algorithm is to combine its basic points with that of algorithms based on Janet bases [12]. We hope to report on our progress in future publications. We also postpone to solve various mathematical problems connected with our algorithm.

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